

# Classifying spaces with virtually cyclic stabilizers for linear groups

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## Abstract

We show that every discrete subgroup of  $GL(n, \mathbb{R})$  admits a finite dimensional classifying space with virtually cyclic stabilizers. Applying our methods to  $SL(3, \mathbb{Z})$ , we obtain a four dimensional classifying space with virtually cyclic stabilizers and a decomposition of the algebraic  $K$ -theory of its group ring.

## 1 Introduction

A classifying space of a discrete group  $\Gamma$  for a family of subgroups  $\mathcal{F}$  is a  $\Gamma$ -CW complex  $X$  with stabilizers in  $\mathcal{F}$  such that  $X^H$  is contractible for every  $H \in \mathcal{F}$ . Such a space is also called a model for  $E_{\mathcal{F}}\Gamma$ . A model for  $E_{\mathcal{F}}\Gamma$  always exists for any given discrete group  $\Gamma$  and a family of subgroups  $\mathcal{F}$ , but it need not be of finite type or finite dimensional (see [17]). The smallest possible dimension of a model for  $E_{\mathcal{F}}\Gamma$  is the geometric dimension of  $\Gamma$  for the family  $\mathcal{F}$ , denoted by  $\text{gd}_{\mathcal{F}}(\Gamma)$ . When  $\mathcal{F}$  is the family of finite, respectively, virtually cyclic subgroups of  $\Gamma$ ,  $E_{\mathcal{F}}\Gamma$  ( $\text{gd}_{\mathcal{F}}(\Gamma)$ ) is denoted by  $\underline{E}\Gamma$  ( $\underline{\text{gd}}(\Gamma)$ ), respectively,  $\underline{\underline{E}}\Gamma$  ( $\underline{\underline{\text{gd}}}(\Gamma)$ ).

For any group  $\Gamma$ , one always has  $\underline{\underline{\text{gd}}}(\Gamma) \leq \underline{\text{gd}}(\Gamma) + 1$  (see [20]). In all examples known so far, a group  $\Gamma$  admits a finite dimensional model for  $\underline{E}\Gamma$  if it admits a finite dimensional model for  $\underline{\underline{E}}\Gamma$ . However, it is still an open problem whether this is always the case. It is known that the invariant  $\underline{\underline{\text{gd}}}(\Gamma)$  can be arbitrarily larger than  $\underline{\text{gd}}(\Gamma)$  (see [8]).

Questions concerning finiteness properties of  $\underline{E}\Gamma$  and  $\underline{\underline{E}}\Gamma$  have been especially motivated by the Farrell–Jones isomorphism conjecture in  $K$ - and  $L$ -theory (see below and [6, 13, 19]). Finding models for  $\underline{\underline{E}}\Gamma$  with good finiteness properties has been proven to be much more difficult than for  $\underline{E}\Gamma$ . So far, such models have been found for polycyclic-by-finite groups [20], word-hyperbolic groups [11], relatively hyperbolic groups [14], countable elementary amenable group of finite Hirsch length [7, 8] and groups acting isometrically with discrete orbits on separable complete  $\text{CAT}(0)$ -spaces, such as mapping class groups and finitely generated linear groups over fields of positive characteristic [9, 18].

Here, we will show that certain subgroups of  $GL(n, \mathbb{C})$  admit a finite dimensional classifying space with virtually cyclic stabilizers.

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**Theorem A.** *Let  $\Gamma$  be a discrete subgroup of  $\mathrm{GL}(n, \mathbb{R})$  such that the Zariski closure of  $\Gamma$  in  $\mathrm{GL}(n, \mathbb{R})$  has dimension  $m$ . Then  $\Gamma$  admits a model for  $\underline{\underline{E}}\Gamma$  of dimension  $m + 1$ .*

Recall that a subgroup  $\Gamma$  of  $\mathrm{GL}(n, \mathbb{C})$  is said to be of *integral characteristic* if the coefficients of the characteristic polynomial of every element of  $\Gamma$  are algebraic integers. It follows that  $\Gamma$  has integral characteristic if and only if the characteristic roots of every element of  $\Gamma$  are algebraic integers (see [2, §2]). The standard embedding  $\Gamma \hookrightarrow \mathrm{GL}(n, \mathbb{C}) \hookrightarrow \mathrm{SL}(n + 1, \mathbb{C})$  allows one to consider  $\Gamma$  as a subgroup of integral characteristic of  $\mathrm{SL}(n + 1, \mathbb{C})$ .

**Theorem B.** *Let  $\Gamma$  be a finitely generated subgroup of  $\mathrm{GL}(n, \mathbb{C})$  of integral characteristic such that there is an upper bound on the Hirsch lengths of its finitely generated unipotent subgroups. Then  $\Gamma$  admits a finite dimensional model for  $\underline{\underline{E}}\Gamma$ .*

**Corollary.** *Let  $\mathbb{F}$  be an algebraic number field and suppose  $\Gamma$  is a subgroup of  $\mathrm{GL}(n, \mathbb{F})$  of integral characteristic. Then  $\Gamma$  admits a finite dimensional model for  $\underline{\underline{E}}\Gamma$ .*

Theorems A and B and the Corollary will be proven in Section 4.

The  $K$ -theoretical Farrell–Jones conjecture (e.g. see [6, 19]) predicts that for a group  $\Gamma$  and a ring  $R$ , the assembly map

$$\mathcal{H}_n^\Gamma(\underline{\underline{E}}\Gamma; \mathbf{K}_R) \rightarrow \mathcal{H}_n^\Gamma(\{*\}; \mathbf{K}) = K_n(R[\Gamma])$$

is an isomorphism for every  $n \in \mathbb{Z}$ . Here  $K_*(R[\Gamma])$  is the algebraic  $K$ -theory of the group ring  $R[\Gamma]$  and  $\mathcal{H}_*^\Gamma(-; \mathbf{K}_R)$  is a generalized equivariant homology theory defined using the  $K$ -theory spectrum  $\mathbf{K}_R$ . This conjecture has been proven for many important classes of groups (and rings), including  $\mathrm{SL}(n, \mathbb{Z})$  when  $R$  is finitely generated as an abelian group (see [4]).

Using the universal property of classifying spaces for families, one can construct a  $\Gamma$ -equivariant inclusion of  $\underline{\underline{E}}\Gamma$  into  $\underline{\underline{E}}\Gamma$ . By a result of Bartels (see [3, Th. 1.3.]) this inclusion induces a split injection  $\mathcal{H}_n^\Gamma(\underline{\underline{E}}\Gamma; \mathbf{K}_R) \rightarrow \mathcal{H}_n^\Gamma(\underline{\underline{E}}\Gamma; \mathbf{K}_R)$ . Hence, there is an isomorphism

$$\mathcal{H}_n^\Gamma(\underline{\underline{E}}\Gamma; \mathbf{K}_R) \cong \mathcal{H}_n^\Gamma(\underline{\underline{E}}\Gamma; \mathbf{K}_R) \oplus \mathcal{H}_n^\Gamma(\underline{\underline{E}}\Gamma, \underline{\underline{E}}\Gamma; \mathbf{K}_R).$$

If  $\Gamma = \mathrm{SL}(3, \mathbb{Z})$ , the term  $\mathcal{H}_n^\Gamma(\underline{\underline{E}}\Gamma; \mathbf{K}_R)$  can be computed using a 3-dimensional cocompact model for  $\underline{\underline{E}}\Gamma$  constructed by Soulé (see [24]). In Theorem, 5.4 we describe the term  $\mathcal{H}_n^\Gamma(\underline{\underline{E}}\Gamma, \underline{\underline{E}}\Gamma; \mathbf{K}_R)$  using a 4-dimensional model for  $\underline{\underline{E}}\Gamma$  we construct in Section 5.

## 2 A push-out construction

A general method to obtain a model for  $\underline{\underline{E}}\Gamma$  from a model for  $\underline{\underline{E}}\Gamma$  is given by Lück and Weiermann in [20, §2]. We will briefly recall this method.

Let  $\Gamma$  be a discrete group and consider the set  $\mathcal{S}$  of all infinite virtually cyclic subgroups of  $\Gamma$ . Two infinite virtually cyclic subgroup of  $\Gamma$  are said to be equivalent if they have infinite intersection in  $\Gamma$ . One easily verifies that this defines an equivalence relation on  $\mathcal{S}$ . If  $H \in \mathcal{S}$ , then its equivalence class will be denoted by  $[H]$ . The set of all equivalence classes of elements of  $\mathcal{S}$  will be denoted by  $[\mathcal{S}]$ . Note

that the conjugation action of  $\Gamma$  on  $\mathcal{S}$  passes to  $[\mathcal{S}]$ . The stabilizer of an  $[H] \in \mathcal{S}$  under this action is the subgroup

$$N_\Gamma[H] = \{g \in \Gamma \mid |H \cap H^g| = \infty\}$$

of  $\Gamma$ . By definition,  $N_\Gamma[H]$  only depends on the equivalence class of  $[H]$  of  $H$ . We may therefore always assume that  $[H]$  is represented by an infinite cyclic group  $H = \langle t \rangle$ . Hence, one can write

$$N_\Gamma[H] = \{g \in \Gamma \mid \exists n, m \in \mathbb{Z} \setminus \{0\} : g^{-1}t^n g = t^m\}.$$

This group is called the commensurator of  $H$  in  $\Gamma$ . Some references, e.g. [8], actually denote this group by  $\text{Comm}_\Gamma[H]$  instead of  $N_\Gamma[H]$ . Note that  $N_\Gamma[H]$  always contains  $H$  as a subgroup.

Let  $\mathcal{S}$  be a complete set of representatives  $[H]$  of the orbits of the conjugation action of  $\Gamma$  on  $[\mathcal{S}]$ . For each  $[H] \in \mathcal{S}$ , let  $\mathcal{F}[H]$  be the family of subgroups of  $N_\Gamma[H]$  containing all finite subgroup of  $N_\Gamma[H]$  and all infinite virtually cyclic subgroup of  $N_\Gamma[H]$  that are equivalent to  $H$ .

**Theorem 2.1** ([20, Theorem 2.3]). *Let*

$$\begin{array}{ccc} \coprod_{[H] \in \mathcal{S}} \Gamma \times_{N_\Gamma[H]} \underline{E}N_\Gamma[H] & \xrightarrow{i} & \underline{E}\Gamma \\ \downarrow \coprod_{[H] \in \mathcal{S}} \text{id}_\Gamma \times f_{[H]} & & \downarrow \\ \coprod_{[H] \in \mathcal{S}} \Gamma \times_{N_\Gamma[H]} E_{\mathcal{F}[H]} N_\Gamma[H] & \longrightarrow & Y \end{array}$$

*be a  $\Gamma$ -equivariant push-out diagram of  $\Gamma$ -CW-complexes such that for each  $[H] \in \mathcal{S}$ , the map  $f_{[H]}$  is cellular and  $N_\Gamma[H]$ -equivariant and  $i$  is a cellular inclusion of  $\Gamma$ -CW-complexes. Then the push-out  $Y$  is a model for  $\underline{\underline{E}}\Gamma$ .*

Using [20, Remark 2.5], one arrives at the following corollary.

**Corollary 2.2** ([20, Remark 2.5]). *If there exists a natural number  $d$  such that for each  $[H] \in \mathcal{S}$*

- $\underline{\text{gd}}(N_\Gamma[H]) \leq d - 1$ ,
- $\underline{\text{gd}}_{\mathcal{F}[H]}(N_\Gamma[H]) \leq d$ ,

*and such that  $\underline{\underline{\text{gd}}}(\Gamma) \leq d$ , then  $\underline{\underline{\text{gd}}}(\Gamma) \leq d$ .*

### 3 On the structure of $N_\Gamma[H]$ in linear groups

We recall that a *real algebraic group* is the set of real points of a linear algebraic group defined over  $\mathbb{R}$ . Throughout this section, we will use some basic facts about (real) algebraic groups for which we refer to [5]. For any subgroup  $K$  of  $\text{GL}(n, \mathbb{R})$ , we will denote the Zariski closure of  $K$  in  $\text{GL}(n, \mathbb{R})$  by  $\overline{K}$ . The notions “connected” and “discrete” will refer to the Hausdorff topology and not to the Zariski topology.

The following result was kindly communicated to us by Herbert Abels.

**Proposition 3.1** (H. Abels). *Let  $G$  be a real algebraic group and suppose  $R$  is its algebraic radical. Suppose  $\Gamma$  is a discrete subgroup of  $G$  such that the  $\pi(\Gamma)$  is Zariski dense in  $G/R$ , where  $\pi : G \rightarrow G/R$  is the natural quotient map. Then  $\pi(\Gamma)$  is discrete.*

*Proof.* Denote by  $C$  the identity component of the closure of the group  $\pi(\Gamma)$  in  $G/R$  in the Hausdorff topology. We need to show that  $C$  is trivial. By Corollary 1.3 of [1], it is solvable. Since  $C$  is normalised by  $\pi(\Gamma)$ , it is also normalised by its Zariski closure  $G/R$ . We obtain that  $C$  is a (Hausdorff and hence Zariski) connected solvable normal subgroup of the semisimple group  $G/R$  and hence it is trivial.  $\square$

Now, let us assume that  $\Gamma$  is a discrete subgroup of  $\mathrm{GL}(n, \mathbb{R})$  and let  $[H]$  be an equivalence class of infinite virtually cyclic subgroups of  $\Gamma$ .

**Lemma 3.2.** *There is a representative  $H \in [H]$  such that  $\overline{H}$  is a Zariski connected abelian normal subgroup of  $\overline{N_\Gamma[H]} \leq \mathrm{GL}(n, \mathbb{R})$  and  $N_\Gamma[H] = N_\Gamma(H)$ , the normaliser of  $H$  in  $\Gamma$ .*

*Proof.* Let  $H \in [H]$ . An algebraic group has only finitely many Zariski connected components. Up to passing to a finite-index subgroup of  $H$  we may therefore assume that  $\overline{H}$  is a Zariski connected algebraic group. Moreover, since  $H$  is abelian, so is  $\overline{H}$ . Let  $x \in N_\Gamma[H]$ . By definition,  $H^x \cap H$  is a finite index subgroup of  $H$ . This implies that  $\overline{H^x \cap H}$  is an algebraic subgroup of  $\overline{H}$  of the same dimension. Because  $\overline{H}$  is Zariski connected, we conclude that  $\overline{H^x \cap H} = \overline{H}$ . Since  $N_\Gamma[H]$  normalizes  $\overline{H}$ , it follows that  $\overline{H}$  is normal in  $\overline{N_\Gamma[H]}$ . From this we deduce that  $\overline{H} \cap N_\Gamma[H]$  is a normal abelian subgroup of  $N_\Gamma[H]$ . Since every discrete subgroup of a finite dimensional abelian Lie group is finitely generated (e.g. see [23, Proposition 3.8]), the structure theorem of finitely generated abelian groups implies that up to passing to a finite-index subgroup,  $H$  is contained in a finite rank free abelian subgroup  $A$  of  $\overline{H} \cap N_\Gamma[H]$  that is normal in  $N_\Gamma[H]$ . But this implies that  $H$  is also normal in  $N_\Gamma[H]$ . Indeed, take  $g \in N_\Gamma[H]$ . Since  $A$  is normal in  $N_\Gamma[H]$ , conjugation by  $g$  induces an automorphism  $\varphi$  of  $A$ . Note that  $H$  has a finite index infinite cyclic overgroup  $H'$  in  $A$  that has a primitive generator  $h$ , meaning that  $h$  is not a proper power of any other element in  $A \cong \mathbb{Z}^r$ . Because  $g \in N_\Gamma[H] = N_\Gamma[H']$ , there exists  $s, t \in \mathbb{Z} \setminus \{0\}$  such that  $s\varphi(h) = th$ . Since  $\varphi(h)$  is also a primitive element, it now follows that  $s$  must divide  $t$  and vice versa. Hence  $s = \pm t$ . It follows that  $H'$  is normal in  $N_\Gamma[H]$  and so is  $H$ .  $\square$

We continue assuming that  $\overline{H}$  is a Zariski connected abelian normal subgroup of  $\overline{N_\Gamma[H]}$  and  $N_\Gamma[H] = N_\Gamma(H)$ . Let  $m$  be the dimension of  $\overline{H}$ .

Recall also that the notion of Hirsch length  $h(S) \in \mathbb{Z}_{\geq 0}$  is defined for all virtually solvable groups  $S$ . The Hirsch length is stable under passing to finite index subgroups. It behaves additively with respect to group extensions of virtually solvable groups and satisfies  $h(\mathbb{Z}) = 1$ . It also satisfies the relation  $h(S) = \sup\{h(S') \mid S' \text{ is a finitely generated subgroup of } S\}$ .

**Proposition 3.3.** *There exists a short exact sequence*

$$1 \rightarrow N \rightarrow N_\Gamma[H]/H \rightarrow Q \rightarrow 1$$

where  $Q$  is a discrete subgroup of a  $k$ -dimensional semisimple algebraic group and  $N$  is a finitely generated solvable group of Hirsch length  $h(N) \leq m - k - 1$ .

*Proof.* Let  $R$  be the algebraic radical of the Zariski closure  $\overline{N_\Gamma[H]} \leq \mathrm{GL}(n, \mathbb{R})$ . There is a short exact sequence

$$1 \rightarrow R \rightarrow \overline{N_\Gamma[H]} \xrightarrow{\pi} S \rightarrow 1$$

where  $S = \overline{N_\Gamma[H]}/R$  is semisimple. Since  $N_\Gamma[H]$  is Zariski dense in  $\overline{N_\Gamma[H]}$  we conclude that  $Q = \pi(N_\Gamma[H])$  is Zariski dense in  $S$ . Hence, by Proposition 3.1 it follows that  $\pi(N_\Gamma[H])$  is a discrete subgroup of the semisimple real algebraic group  $S$ . Since the Zariski connected abelian normal subgroup  $\overline{H}$  of  $\overline{N_\Gamma[H]}$  has finitely many Hausdorff connected components, up to passing to a finite-index subgroup, we may assume that  $H$  is contained in  $R$ . Denoting  $N = (R \cap N_\Gamma[H])/H$ , we obtain a short exact sequence

$$1 \rightarrow N \rightarrow N_\Gamma[H]/H \rightarrow Q \rightarrow 1.$$

Since every discrete subgroup of a connected solvable Lie group is finitely generated (see [23, Proposition 3.8]) and  $R$ , being an algebraic group, has finitely many connected components, it follows that  $N$  is a finitely generated solvable group. Suppose  $S$  had dimension  $k$ . Then  $R$  has dimension  $m - k$ . The dimension of  $R$  is an upper bound for  $\text{gd}(R \cap N_\Gamma[H])$  (e.g. see [17, Theorem 4.4]). Moreover,  $\text{gd}(R \cap N_\Gamma[H])$  is bounded from below by the Hirsch length of  $R \cap N_\Gamma[H]$ , since the Hirsch length of a solvable group coincides with its rational homological dimension (see [26]). It follows that  $h(R \cap N_\Gamma[H]) \leq m - k$ . Hence, the Hirsch length of  $N$  is at most  $m - k - 1$ .  $\square$

## 4 The proofs of the main theorems

We are now ready to prove Theorems A and B and their Corollary.

*Proof of Theorem A.* Since the dimension of the Zariski closure of  $\Gamma$  in  $\text{GL}(n, \mathbb{R})$  is an upper bound for  $\text{gd}(\Gamma)$  (e.g. see [17, Theorem 4.4]), we have  $\text{gd}(N_\Gamma[H]) \leq \text{gd}(\Gamma) \leq m$  for every infinite cyclic subgroup  $H$  of  $\Gamma$ . Now fix  $[H] \in \mathcal{S}$  and consider the exact sequence

$$1 \rightarrow N \rightarrow N_\Gamma[H]/H \rightarrow Q \rightarrow 1$$

resulting from Proposition 3.3. By [8, Lemma 4.2], we have  $\text{gd}_{\mathcal{S}[H]}(N_\Gamma[H]) = \text{gd}(N_\Gamma[H]/H)$  for every  $[H] \in \mathcal{S}$ . If the Hirsch length of  $N$  is at most 1, then since  $N$  is finitely generated it follows that  $N$  is virtually cyclic. So, every finite extension  $T$  of  $N$  must be virtually cyclic as well. In this case one has  $\text{gd}(T) \leq 1$ . If the Hirsch length of  $N$  is at least 2, then it follows from [10, Corollary 4] that every finite extension  $T$  of  $N$  has  $\text{gd}(T) \leq h + 1$ . Since  $\text{gd}(S) \leq k$  by [17, Theorem 4.4], it follows from [8, Corollary 2.3] that  $\text{gd}_{\mathcal{S}[H]}(N_\Gamma[H]) \leq (m - k - 1) + 1 + k = m$  for every  $[H] \in \mathcal{S}$ . The theorem now follows from Corollary 2.2.  $\square$

*Proof of Theorem B.* We may assume that  $\Gamma$  is a subgroup of  $\text{SL}(n, \mathbb{C})$  of integral characteristic. Let  $A$  be the finitely generated unital subring of  $\mathbb{C}$  generated by the matrix entries of a finite set of generators of  $\Gamma$  and their inverses. Then  $\Gamma$  is a subgroup of  $\text{SL}(n, A)$ .

Let  $\mathbb{F}$  denote the quotient field of  $A$ . Proceeding as in the proof of Theorem 3.3 of [2], we obtain an epimorphism  $\rho : \Gamma \rightarrow H_1 \times \cdots \times H_r$  such that the kernel  $U$  of  $\rho$  is a unipotent subgroup of  $H$  and for each  $1 \leq i \leq r$ ,  $H_i$  is a subgroup of some  $\text{GL}(n_i, \mathbb{F})$  of integral characteristic where the canonical action of  $H_i$  on  $\mathbb{F}^{n_i}$  is irreducible. Following the proof of Proposition 2.3 of [2], we have that for each subgroup  $H_i$ , there exists a finite field extension  $L_i$  of  $\mathbb{Q}$  such that  $H_i$  is isomorphic to a subgroup  $H'_i$  of some  $\text{GL}(m_i, L_i)$ , which is absolutely irreducible and of integral characteristic. Now, according to the proof of Proposition 2.1 of

[2], each  $H'_i$  embeds as a discrete subgroup of  $\mathrm{GL}(m_i, \mathbb{R})^{r_i} \times \mathrm{GL}(m_i, \mathbb{C})^{s_i}$  for some nonnegative integers  $r_i$  and  $s_i$ . So, by Theorem A, we have that  $\underline{\underline{\mathrm{gd}}}(H_1 \times \cdots \times H_r) < \infty$ . Applying Corollary 6.1 of [8], we obtain that

$$\underline{\underline{\mathrm{gd}}}(\Gamma) \leq \underline{\underline{\mathrm{gd}}}(H_1 \times \cdots \times H_r) + h + 3$$

where  $h$  is the Hirsch length of  $U$ .  $\square$

*Proof of Corollary.* As in the proof of Theorem B,  $\Gamma$  fits into an extension

$$1 \rightarrow U \rightarrow \Gamma \rightarrow H_1 \times \cdots \times H_r \rightarrow 1$$

where  $U$  is a unipotent subgroup and for each  $1 \leq i \leq r$ ,  $H_i$  is a subgroup of integral characteristic of some  $\mathrm{GL}(n_i, \mathbb{F})$  such that the canonical action of  $H_i$  on  $\mathbb{F}^{n_i}$  is irreducible. Following the proof of Proposition 2.1 of [2], we obtain that each  $H_i$  is isomorphic to a discrete subgroup of  $\mathrm{GL}(m_i, \mathbb{R})^{r_i} \times \mathrm{GL}(m_i, \mathbb{C})^{s_i}$  for some positive integers  $r_i$  and  $s_i$ . By considering the upper central series of the subgroup of strictly upper triangular matrices  $\mathrm{Tr}(n, \mathbb{F})$  of  $\mathrm{GL}(n, \mathbb{F})$  and noticing that the additive group of  $\mathbb{F}$  has finite Hirsch length because it is isomorphic to a finite direct product of copies of  $(\mathbb{Q}, +)$ , it follows that  $\mathrm{Tr}(n, \mathbb{F})$  has finite Hirsch length. Since the subgroup  $U$  of  $\Gamma$  is conjugate in  $\mathrm{GL}(n, \mathbb{C})$  to a subgroup of  $\mathrm{Tr}(n, \mathbb{F})$ , it also has finite Hirsch length. Just as in the proof of Theorem B, we now conclude that  $\underline{\underline{\mathrm{gd}}}(\Gamma) < \infty$ .  $\square$

## 5 The case of $\mathrm{SL}(3, \mathbb{Z})$

Consider the group  $\mathrm{SL}(3, \mathbb{Z})$  and let  $\mathcal{S}$  be a set of representatives of the orbits of the conjugation action of  $\mathrm{SL}(3, \mathbb{Z})$  on the set of equivalence classes of infinite virtually cyclic subgroups of  $\mathrm{SL}(3, \mathbb{Z})$  (see Section 2). We note that the infinite virtually cyclic subgroups of  $\mathrm{SL}(3, \mathbb{Z})$  are listed, up to isomorphism, in [25] and [27]. From this classification it follows that every infinite virtually cyclic subgroup  $V$  of  $\Gamma$  that is not isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z} \oplus \mathbb{Z}_2$  fits into a short exact sequence

$$1 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \rightarrow V \rightarrow \mathbb{Z}_2 \rightarrow 1.$$

**Definition 5.1.** We define the following subsets of  $\mathcal{S}$ .

- (a) The set  $\mathcal{S}_1$  contains all  $[H] \in \mathcal{S}$  such that  $[H]$  has a representative whose generator has two complex conjugate eigenvalues and one real eigenvalue different from 1;
- (b) The set  $\tilde{\mathcal{S}}_1$  contains all  $[H] \in \mathcal{S}$  such that  $[H]$  has a representative whose generator has exactly one eigenvalue that is a root of unity.
- (c) The set  $\mathcal{S}_2$  contains all  $[H] \in \mathcal{S}$  such that  $[H]$  has a representative with a generator all of whose eigenvalues are real and not equal to  $\pm 1$ ;
- (d) The set  $\tilde{\mathcal{S}}_2$  contains all  $[H] \in \mathcal{S}$  such that  $[H]$  has a representative with a generator all of whose eigenvalues equal 1 and which cannot be conjugated into the center of the strictly upper triangular matrices in  $\mathrm{SL}(3, \mathbb{Z})$ .
- (e) The set  $\mathcal{S}_3$  contains all  $[H]$  such that  $[H]$  has a representative with a generator that can be conjugated into the center of the strictly upper triangular matrices in  $\mathrm{SL}(3, \mathbb{Z})$ .

**Lemma 5.2.** *One can write  $\mathcal{J}$  as a disjoint union*

$$\mathcal{J} = \mathcal{J}_1 \sqcup \tilde{\mathcal{J}}_1 \sqcup \mathcal{J}_2 \sqcup \tilde{\mathcal{J}}_2 \sqcup \mathcal{J}_3$$

*and the set  $\mathcal{J}_3$  contains exactly one element.*

*Proof.* This is left as an easy exercise to the reader.  $\square$

The group  $\Gamma = \mathrm{SL}(3, \mathbb{Z})$  is a discrete subgroup of  $\mathrm{GL}(3, \mathbb{R})$ . Hence, we know from Lemma 3.2 that for every equivalence class  $[H]$  of infinite virtually cyclic subgroups of  $\Gamma$ , there exists a representative  $H$  such that  $N_\Gamma[H] = N_\Gamma(H)$ . Using this fact, we will now determine for each  $[H] \in \mathcal{J}$  the structure of the group  $N_\Gamma[H]$ .

**Lemma 5.3.** *For each  $[H] \in \mathcal{J}$ , the following holds.*

- (a) *If  $[H] \in \mathcal{J}_1$ , then  $N_\Gamma[H] \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ .*
- (b) *If  $[H] \in \tilde{\mathcal{J}}_1$ , then  $N_\Gamma[H]$  has a subgroup of index at most two isomorphic to  $\mathbb{Z}_2 \oplus \mathbb{Z}$ .*
- (c) *If  $[H] \in \mathcal{J}_2$ , then  $N_\Gamma[H] \cong \mathbb{Z}_2 \oplus \mathbb{Z}^2$ .*
- (d) *If  $[H] \in \tilde{\mathcal{J}}_2$ , then  $N_\Gamma[H]$  has a subgroup of index at most two isomorphic to  $\mathbb{Z}^2$ .*
- (e) *If  $[H] \in \mathcal{J}_3$ , then  $N_\Gamma[H]$  is isomorphic to  $\mathrm{Tr}(3, \mathbb{Z}) \rtimes_\varphi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ , where*

$$\mathrm{Tr}(3, \mathbb{Z}) = \langle x, y, z \mid [x, y] = z, [x, z] = e, [y, z] = e \rangle$$

*is isomorphic to the group of strictly upper triangular integral matrices,*

$$\varphi((1, 0))(x) = x^{-1}, \varphi((1, 0))(y) = y^{-1}, \varphi((1, 0))(z) = z$$

*and*

$$\varphi((0, 1))(x) = x^{-1}, \varphi((0, 1))(y) = y, \varphi((0, 1))(z) = z^{-1}.$$

*Proof.* Take  $[H] \in \mathcal{J}$  and let  $A \in \Gamma$  be an infinite order matrix such that  $N_\Gamma[H] = N_\Gamma(H)$ , where  $\langle A \rangle = H$ . Note that we may replace  $A$  by a power of  $A$  in order to assume that  $A$  does not have any eigenvalues that are non-trivial roots of unity.

First assume that  $[H] \in \mathcal{J}_1 \sqcup \mathcal{J}_2$ . This means that all eigenvalues of  $A$  are different from 1. The characteristic polynomial  $p(x)$  of  $A$  is therefore irreducible over  $\mathbb{Q}$ . Indeed, if  $p(x)$  was reducible over  $\mathbb{Q}$ , then  $A$  would have a rational eigenvalue  $\mu$ . But since  $A \in \mathrm{SL}(3, \mathbb{Z})$ , it follows from the rational root theorem that  $\mu = \pm 1$ , which is a contradiction. Also note that the normalizer of  $H$  must equal the centralizer of  $H$ . Indeed, an element of the normalizer of  $H$  that does not commute with  $A$  must send a eigenvector of  $A$  with eigenvalue  $\mu$  to an eigenvector of  $A$  with eigenvalue  $\mu^{-1}$ , which would imply that  $A$  has an eigenvalue equal to 1. As illustrated for example in [12, Prop. 3.7] and [22, section 4], an application of the Dirichlet unit theorem shows that the centralizer  $C_\Gamma(H)$  of  $A$  in  $\Gamma$  equals  $\mathbb{Z}^{r+s-1} \oplus \mathbb{Z}_2$ , where  $r$  is the number of real roots of  $p(x)$  and  $2s$  is the number of complex roots of  $p(x)$ . Hence, if all eigenvalues of  $A$  are real then  $N_\Gamma(H) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$  and if  $A$  has two complex conjugate eigenvalues then  $N_\Gamma(H) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ . This proves (a) and (c).

Secondly, assume that  $[H] \in \tilde{\mathcal{H}}_1$ . Then  $A$  has exactly one eigenvalue equal to 1. Hence,  $A$  is conjugate in  $\mathrm{SL}(3, \mathbb{Z})$  to a matrix of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & & \\ 0 & M & \end{bmatrix}.$$

Since we are only interested in the structure of the normalizer of  $H$  up to isomorphism, we may as well assume that  $A$  is of this form. If a matrix  $B \in \mathrm{SL}(3, \mathbb{Z})$  commutes with  $A$  it must preserve the 1-dimensional eigenspace of  $A$  with eigenvalue 1. Therefore, this is also an eigenspace of  $B$ , with eigenvalue  $\pm 1$ . We conclude that  $B$  must be of the form

$$B = \begin{bmatrix} \pm 1 & x & y \\ 0 & & \\ 0 & N & \end{bmatrix}.$$

By elementary matrix computations, one checks that such a matrix  $B$  commutes with  $A$  if and only if  $M$  commutes with  $N$  and

$$(M^t - \mathrm{Id}) \begin{bmatrix} x \\ y \end{bmatrix} = (N^t - \mathrm{Id}) \begin{bmatrix} a \\ b \end{bmatrix}.$$

Since  $(M^t - \mathrm{Id})$  is an invertible matrix,  $B$  is completely determined by  $N$  and the fact that it commutes with  $A$ . We therefore obtain an isomorphism  $C_\Gamma(A) = C_{\mathrm{GL}(2, \mathbb{Z})}(M)$ . By analyzing centralizers in  $\mathrm{GL}(2, \mathbb{Z})$ , for example using the Dirichlet unit theorem, it follows that the centralizer  $C_\Gamma(A)$  is  $\mathbb{Z}_2 \oplus \mathbb{Z}$ . This proves (b).

Finally, assume that all eigenvalues of  $A$  equal 1. In this case  $A$  can be conjugated inside  $\mathrm{SL}(3, \mathbb{Z})$  to a strictly upper triangular matrix. Hence, we may again assume that  $A$  is a strictly upper triangular matrix. If  $A$  is of the form

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

where  $a$  and  $b$  are both non-zero, then one may check by elementary matrix computations that a matrix  $B \in \mathrm{SL}(3, \mathbb{Z})$  commutes with  $A$  if and only if it is of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

where  $ay - bx = 0$ . This shows that in this case the centralizer of  $H$  in  $\Gamma$  is isomorphic to  $\mathbb{Z}^2$ , and hence the normalizer  $N_\Gamma(H)$  has a subgroup of index at most two isomorphic to  $\mathbb{Z}^2$ . If on the other hand,  $A$  is of the form (1) where  $ab = 0$  then  $A$  can be conjugated in  $\mathrm{SL}(3, \mathbb{Z})$  to matrix of the form (1) where  $a$  and  $b$  are both zero. In this case the centralizer  $C_\Gamma(H)$  is isomorphic to the group

$$\left\{ \begin{bmatrix} \pm 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & \pm 1 \end{bmatrix} \mid x, y, z \in \mathbb{Z} \right\}.$$



One can now easily verify via explicit matrix computation that the normalizer  $N_\Gamma(H)$  is isomorphic to a semi-direct product of  $C_\Gamma(H)$  with

$$\left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

It follows that  $N_\Gamma(H)$  is isomorphic to the semi-direct product

$$\mathrm{Tr}(3, \mathbb{Z}) \rtimes_\varphi (\mathbb{Z}_2 \oplus \mathbb{Z}_2).$$

□

In [24], a 3-dimensional model  $X$  for  $\underline{E}\Gamma$  is constructed. This model has the property that the orbit-space  $\Gamma \backslash X = \underline{B}\Gamma$  is contractible. Moreover, this model is of minimal dimension since  $\Gamma$  contains the strictly upper triangular matrices  $\mathrm{Tr}(3, \mathbb{Z})$ , which has cohomological dimension 3. Since for each  $[H] \in \mathcal{J}$ ,  $N_\Gamma[H]$  is either virtually- $\mathbb{Z}$ , virtually- $\mathbb{Z}^2$  or virtually- $\mathrm{Tr}(3, \mathbb{Z})$  by the lemma above, a model for  $\underline{E}N_\Gamma[H]$  can be chosen to be either  $\mathbb{R}$ ,  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , respectively (see, e.g., [17, Ex. 5.26]). Moreover,  $H$  can be chosen to be normal in  $N_\Gamma[H]$  in which case a model for  $E_{\mathcal{J}[H]}N_\Gamma[H]$  is given by a model for  $\underline{E}N_\Gamma[H]/H$ , where the action is obtained via the projection  $N_\Gamma[H] \rightarrow N_\Gamma[H]/H$ . Hence, a model for  $E_{\mathcal{J}[H]}N_\Gamma[H]$  can be chosen to be either  $\{*\}$ ,  $\mathbb{R}$  or  $\mathbb{R}^2$ , respectively. Using the universal property of classifying spaces for families, one obtains a cellular  $\Gamma$ -equivariant map

$$f : \coprod_{[H] \in \mathcal{J}} \Gamma \times_{N_\Gamma[H]} \underline{E}N_\Gamma[H] \rightarrow X$$

Note that the mapping cylinder  $M_f$  of  $f$  is a 4-dimensional model for  $\underline{E}\Gamma$ , since  $\underline{E}N_\Gamma[H]$  is at most 3-dimensional and  $M_f$  is  $\Gamma$ -homotopy equivalent to  $X$ . We obtain an equivariant cellular inclusion

$$i : \coprod_{[H] \in \mathcal{J}} \Gamma \times_{N_\Gamma[H]} \underline{E}N_\Gamma[H] \rightarrow M_f.$$

Using  $i$  and the models for  $E_{\mathcal{J}[H]}N_G[H]$  described above, one can construct a  $\Gamma$ -equivariant push-out diagram that by Theorem 2.1 produces a 4-dimensional model  $Y$  for  $\underline{E}\Gamma$ . We claim that this model is of minimal dimension. Indeed, take  $\Gamma$ -orbits of the push-out diagram constructed above and consider the long exact Mayer-Vietoris cohomology sequence with  $\mathbb{Q}$ -coefficients obtained from the resulting push-out diagram. This leads to the exact sequence

$$H^3(\underline{B}\Gamma, \mathbb{Q}) \rightarrow H^3(\underline{B}N_\Gamma[H], \mathbb{Q}) \rightarrow H^4(\underline{B}\Gamma, \mathbb{Q}) \rightarrow 0,$$

where  $[H] \in \mathcal{J}_3$ . Since  $\underline{B}\Gamma$  is contractible and  $N_\Gamma[H] \cong \mathrm{Tr}(3, \mathbb{Z}) \rtimes_\varphi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$  by the lemma above, we obtain an isomorphism

$$H^3(\mathrm{Tr}(3, \mathbb{Z}) \rtimes_\varphi (\mathbb{Z}_2 \oplus \mathbb{Z}_2), \mathbb{Q}) \cong H^4(\underline{B}\Gamma, \mathbb{Q}).$$

As we are working with  $\mathbb{Q}$ -coefficients, an application of the Lyndon-Hochschild-Serre spectral sequence tells us that

$$H^3(\mathrm{Tr}(3, \mathbb{Z}) \rtimes_\varphi (\mathbb{Z}_2 \oplus \mathbb{Z}_2), \mathbb{Q}) \cong H^3(\mathrm{Tr}(3, \mathbb{Z}), \mathbb{Q})^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}.$$

Moreover, since  $\text{Tr}(3, \mathbb{Z})$  fits into the central extension

$$1 \rightarrow \mathbb{Z} \cong \langle z \rangle \rightarrow \text{Tr}(3, \mathbb{Z}) \rightarrow \mathbb{Z}^2 \cong \langle x, y \rangle \rightarrow 0,$$

another application of the Lyndon–Hochschild–Serre spectral sequence yields

$$H^3(\text{Tr}(3, \mathbb{Z}), \mathbb{Q})^{\mathbb{Z}_2 \oplus \mathbb{Z}_2} \cong H^2(\langle x, y \rangle, H^1(\langle z \rangle, \mathbb{Q}))^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}.$$

Using the explicit description of the map  $\varphi : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow \text{Aut}(\text{Tr}(3, \mathbb{Z}))$  in the lemma above, and the fact that

$$H^2(\langle x, y \rangle, H^1(\langle z \rangle, \mathbb{Q})) = \text{Hom}(\Lambda^2(\langle x, y \rangle), \text{Hom}(\Lambda^1(\langle z \rangle), \mathbb{Q})) \cong \mathbb{Q},$$

one checks that the action of  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  on  $H^2(\langle x, y \rangle, H^1(\langle z \rangle, \mathbb{Q}))$  is trivial. We conclude that  $H^4(\underline{\underline{B}}\Gamma, \mathbb{Q}) \cong \mathbb{Q}$ , proving that there cannot exist a model for  $\underline{\underline{E}}\Gamma$  of dimension strictly smaller than 4.

As mentioned in the introduction, for  $\Gamma = \text{SL}(3, \mathbb{Z})$ , the Farrell–Jones conjecture implies that for any ring  $R$  that is finitely generated as an abelian group, one has

$$K_n(R[\Gamma]) \cong \mathcal{H}_n^\Gamma(\underline{E}\Gamma; \mathbf{K}_R) \oplus \mathcal{H}_n^\Gamma(\underline{\underline{E}}\Gamma, \underline{E}\Gamma; \mathbf{K}_R)$$

for every  $n \in \mathbb{Z}$ . Using the model  $Y$  for  $\underline{\underline{E}}\Gamma$  constructed above and Lemma 5.3 we obtain a description of the term  $\mathcal{H}_n^\Gamma(\underline{\underline{E}}\Gamma, \underline{E}\Gamma; \mathbf{K}_R)$ . We summarize this description in the following theorem. Note that given a  $\Gamma$ -map  $f : X \rightarrow Y$ , the homology group  $\mathcal{H}_n^\Gamma(Y, X; \mathbf{K}_R)$  is by definition the relative homology group  $\mathcal{H}_n^\Gamma(M_f, X; \mathbf{K}_R)$ , where  $M_f$  is the mapping cylinder of  $f$ .

**Theorem 5.4.** *Let  $\Gamma = \text{SL}(3, \mathbb{Z})$  and let  $R$  be a ring that is finitely generated as an abelian group. Then,*

$$K_n(R[\Gamma]) \cong \mathcal{H}_n^\Gamma(\underline{E}\Gamma; \mathbf{K}_R) \oplus \mathcal{H}_n(\mathcal{J}_1) \oplus \mathcal{H}_n(\tilde{\mathcal{J}}_1) \oplus \mathcal{H}_n(\mathcal{J}_2) \oplus \mathcal{H}_n(\tilde{\mathcal{J}}_2) \oplus \mathcal{H}_n(\mathcal{J}_3)$$

where,

- (a) for  $[H] \in \mathcal{J}_1$ ,  $N_\Gamma[H] \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ ,  $\underline{\underline{E}}N_\Gamma[H] = \{*\}$ ,  $\underline{E}N_\Gamma[H] = \mathbb{R}$  and

$$\mathcal{H}_n(\mathcal{J}_1) = \bigoplus_{[H] \in \mathcal{J}_1} \mathcal{H}_n^{N_\Gamma[H]}(\{*\}, \mathbb{R}; \mathbf{K}_R),$$

- (b) for  $[H] \in \tilde{\mathcal{J}}_1$ ,  $N_\Gamma[H]$  has a subgroup of index at most two isomorphic to  $\mathbb{Z}_2 \oplus \mathbb{Z}$ ,  $\underline{\underline{E}}N_\Gamma[H] = \{*\}$ ,  $\underline{E}N_\Gamma[H] = \mathbb{R}$  and

$$\mathcal{H}_n(\tilde{\mathcal{J}}_1) = \bigoplus_{[H] \in \tilde{\mathcal{J}}_1} \mathcal{H}_n^{N_\Gamma[H]}(\{*\}, \mathbb{R}; \mathbf{K}_R),$$

- (c) for  $[H] \in \mathcal{J}_2$ ,  $N_\Gamma[H] \cong \mathbb{Z}_2 \oplus \mathbb{Z}^2$ ,  $E_{\mathcal{J}[H]}N_\Gamma[H] = \underline{\underline{E}}N_\Gamma[H]/H = \mathbb{R}$ ,  $\underline{E}N_\Gamma[H] = \mathbb{R}^2$  and

$$\mathcal{H}_n(\mathcal{J}_2) = \bigoplus_{[H] \in \mathcal{J}_2} \mathcal{H}_n^{N_\Gamma[H]}(\mathbb{R}, \mathbb{R}^2; \mathbf{K}_R),$$

- (d) for  $[H] \in \tilde{\mathcal{J}}_2$ ,  $N_\Gamma[H]$  has a subgroup of index at most two isomorphic to  $\mathbb{Z}^2$ ,  $E_{\mathcal{J}[H]}N_\Gamma[H] = \underline{E}N_\Gamma[H]/H = \mathbb{R}$ ,  $\underline{E}N_\Gamma[H] = \mathbb{R}^2$  and

$$\mathcal{H}_n(\tilde{\mathcal{J}}_2) = \bigoplus_{[H] \in \tilde{\mathcal{J}}_2} \mathcal{H}_n^{N_\Gamma[H]}(\mathbb{R}, \mathbb{R}^2; \mathbf{K}_R),$$

- (e) for  $[H] \in \tilde{\mathcal{J}}_3$ ,  $N_\Gamma[H] \cong \mathrm{Tr}(3, \mathbb{Z}) \rtimes_\varphi (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ ,  $E_{\mathcal{J}[H]}N_\Gamma[H] = \underline{E}N_\Gamma[H]/H = \mathbb{R}^2$ ,  $\underline{E}N_\Gamma[H] = \mathbb{R}^3$  and

$$\mathcal{H}_n(\mathcal{J}_3) = \mathcal{H}_n^{N_\Gamma[H]}(\mathbb{R}^2, \mathbb{R}^3; \mathbf{K}_R).$$

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